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Discrete and ultradiscrete Bäcklund transformation for KdV equation

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Abstract
The Bäcklund transformation for the discrete Korteweg–de Vries equation is introduced in the bilinear form. The superposition formula is also derived from the transformation. An ultradiscrete analogue of the transformation is presented by means of the ultradiscretization technique. This analogue gives the Bäcklund transformation for the box and ball system. The ultradiscrete soliton solutions for the system are also discussed with explicit examples.

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1. Introduction

The Bäcklund transformation has played an important role in the development of soliton theory. In the case of the Korteweg–de Vries (KdV) equation,

$$w_t - 3{w_x}^2 + w_{xxx} = 0, \tag{1}$$

the Bäcklund transformation between two solutions, w_1 and w_2, is given by

$$(w_1 + w_2)_x = 2\lambda + \tfrac{1}{2}(w_1 - w_2)^2 \tag{2a}$$

$$(w_1 - w_2)_t = 3\left({w_{1x}}^2 - {w_{2x}}^2\right) - (w_1 - w_2)_{xxx}, \tag{2b}$$

where λ is a parameter. Integrating (2) with a given $(N-1)$-soliton solution w_1 and λ, we obtain the N-soliton solution w_2 in principle. Furthermore, there exists 'superposition formula' [1]

$$w_{12} = w_0 - \frac{4(\lambda_1 - \lambda_2)}{w_1 - w_2}, \tag{3}$$

which is an algebraic relation among four solutions, w_0, w_1, w_2 and w_{12}. In the case of the soliton solution, if w_0 is the $(N-1)$-soliton solution and w_i $(i = 1, 2)$ are the N-soliton solutions, w_{12} is the $(N+1)$-soliton solution. The Bäcklund transformation in the bilinear form is also presented by Hirota [2]. The bilinear form gives us a clear viewpoint in the structure of solutions written in terms of the Wronskian [3, 4]. The bilinearizing technique (see, for example, [5]) also clarifies the relationship among the Bäcklund transformation, the Miura transformation [6] and the inverse scattering method [7].

Discrete analogues of the Bäcklund transformation are also discussed for the discrete KdV (dKdV) equation. Hirota presents the discrete Bäcklund transformation in terms of the bilinear form [8]. Nijhoff *et al* give it in the context of the inverse scattering setting (see, for example, [9]). Schiff studies the discrete KdV equation by means of loop group methods and presents a discrete analogue of the superposition formula [10].

Cellular automaton (CA) is a discrete dynamical system which consists of a regular array of cells. Each cell takes a finite number of states updated by a given rule in discrete time steps. Although the updating rule is usually simple, CAs may capture the essential mechanisms for many physical, social or biological phenomena (see, for example, [11]). Moreover, CAs are suitable for computer experiments since all variables take discrete values. Ultradiscretization [12] is a procedure transforming a given discrete equation into a CA (or an ultradiscrete system). In general, to apply this procedure, we first replace a dependent variable a in a given equation with a new variable A by $a = \mathrm{e}^{A/\varepsilon}$ upon introduction of a parameter $\varepsilon > 0$. Then in the limit $\varepsilon \to +0$, addition, multiplication and division of the original variables are replaced with $\max[*, *]$, addition and subtraction for the new ones, respectively. It is an interesting problem to study the structure preserved in ultradiscretization of an integrable system by means of this procedure. The ultradiscrete analogue of the KdV equation is known as the soliton cellular automaton [13], or the box and ball system (BBS) [14]. In [15], we propose an ultradiscrete analogue of the Miura transformation and report its connection with the box and ball system with a carrier [16]. Shinzawa and Hirota discuss an ultradiscrete analogue of the Bäcklund transformation for the discrete Kadomtsev–Petviashvili (dKP) equation and give the ultradiscrete one-, two- and three-soliton solutions in *a priori* way [17], while they do not consider explicit relationship between the ultradiscrete solutions and those of the dKP equation.

In section 2, we introduce the Bäcklund transformation for the dKdV equation and show the way of constructing soliton solutions through the transformation. We also present a superposition formula for the dKdV equation which is a different form from the one in [10]. Then in section 3 we construct the ultradiscrete analogue of the Bäcklund transformation and present the ultradiscrete soliton solutions. We also discuss the connection between the discrete soliton solutions and the ultradiscrete ones. Finally, concluding remarks are given in section 4.

2. Discrete system

The dKdV equation is written in the bilinear form

$$f_j^{t-1} f_{j-1}^{t-1} + \gamma f_{j-1}^{t} f_j^{t-2} = (1+\gamma) f_{j-1}^{t-2} f_j^{t}, \tag{4}$$

where γ is a parameter. Its N-soliton solution is given in terms of the determinant of size $N \times N$ [18],

$$\begin{tikzcd} & g_j^t \arrow[dr, "k_{\mathrm{II}}"] & \\ f_j^t \arrow[ur, "k_{\mathrm{I}}"] \arrow[dr, "k_{\mathrm{II}}"'] & & h_j^t \\ & \tilde{g}_j^t \arrow[ur, "k_{\mathrm{I}}"'] & \end{tikzcd}$$

Figure 1. Superposition diagram.

$$f_j^t = \begin{vmatrix} \dfrac{c_1 {k_1}^{2(N-1)}}{X_1} + \dfrac{d_1 X_1}{{k_1}^{2(N-1)}} & \cdots & \dfrac{c_1 {k_1}^{2(N-l)}}{X_1} + \dfrac{d_1 X_1}{{k_1}^{2(N-l)}} & \cdots & \dfrac{c_1}{X_1} + d_1 X_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{c_N {k_N}^{2(N-1)}}{X_N} + \dfrac{d_N X_N}{{k_N}^{2(N-1)}} & \cdots & \dfrac{c_N {k_N}^{2(N-l)}}{X_N} + \dfrac{d_N X_N}{{k_N}^{2(N-l)}} & \cdots & \dfrac{c_N}{X_N} + d_N X_N \end{vmatrix}, \tag{5}$$

where c_i, d_i, k_i $(i = 1, 2, \ldots, N)$ are parameters and $X_i := {k_i}^j {\omega_i}^t$. Here, ω_i is a parameter satisfying the dispersion relation

$$\gamma = \frac{1 - {k_i}^2 {\omega_i}^2}{\left({k_i}^2 - 1\right)\left({\omega_i}^2 + 1\right)}. \tag{6}$$

Note that we may put $c_i = 1/d_i$ or $c_i = -1/d_i$ without loss of generality. The Bäcklund transformation of the dKdV equation may be written as

$$\begin{aligned} (k_N + 1/k_N)(\omega_N + 1/\omega_N)(1+\gamma) f_j^t g_{j-1}^{t-1} &= (\omega_N + 1/\omega_N)(1 + 2\gamma) f_{j-1}^t g_j^{t-1} \\ &\quad + (k_N + 1/k_N) f_j^{t-1} g_{j-1}^t, \end{aligned} \tag{7a}$$

$$\begin{aligned} (k_N + 1/k_N)(\omega_N + 1/\omega_N)(1+\gamma) f_{j-1}^{t-1} g_j^t &= (k_N + 1/k_N) f_{j-1}^t g_j^{t-1} \\ &\quad + (\omega_N + 1/\omega_N)(1 + 2\gamma) f_j^{t-1} g_{j-1}^t. \end{aligned} \tag{7b}$$

If we take the $(N-1)$-soliton solution (5) of size $(N-1) \times (N-1)$ for g_j^t in (7), then f_j^t becomes the N-soliton solution (5) of size $N \times N$.

Let us study the superposition formula in the discrete system. For this purpose, we consider the diagram in figure 1. Under this setting, we have the eight relations, namely, (7) for f_j^t, g_j^t with k_{I}, for $f_j^t, \tilde{g}_j^t$ with k_{II}, for g_j^t, h_j^t with k_{II} and for $\tilde{g}_j^t, h_j^t$ with k_{I}. By eliminating the dependent variables with the argument $t-1$ from these relations, we obtain a superposition formula

$$\rho = \rho_0 \frac{(k_{\mathrm{I}} + 1/k_{\mathrm{I}})\rho_{\mathrm{I}} - (k_{\mathrm{II}} + 1/k_{\mathrm{II}})\rho_{\mathrm{II}}}{(k_{\mathrm{II}} + 1/k_{\mathrm{II}})\rho_{\mathrm{I}} - (k_{\mathrm{I}} + 1/k_{\mathrm{I}})\rho_{\mathrm{II}}}, \tag{8}$$

where $\rho_0 := f_{j-1}^t \big/ f_j^t$, $\rho_{\mathrm{I}} := g_{j-1}^t \big/ g_j^t$, $\rho_{\mathrm{II}} := \tilde{g}_{j-1}^t \big/ \tilde{g}_j^t$ and $\rho := h_{j-1}^t \big/ h_j^t$. Moreover, by eliminating the dependent variables with the argument $j-1$, we have another superposition formula

$$\sigma = \sigma_0 \frac{(\omega_{\mathrm{I}} + 1/\omega_{\mathrm{I}})\sigma_{\mathrm{I}} - (\omega_{\mathrm{II}} + 1/\omega_{\mathrm{II}})\sigma_{\mathrm{II}}}{(\omega_{\mathrm{II}} + 1/\omega_{\mathrm{II}})\sigma_{\mathrm{I}} - (\omega_{\mathrm{I}} + 1/\omega_{\mathrm{I}})\sigma_{\mathrm{II}}}, \tag{9}$$

where $\sigma_0 := f_j^{t-1} \big/ f_j^t$, $\sigma_{\mathrm{I}} := g_j^{t-1} \big/ g_j^t$, $\sigma_{\mathrm{II}} := \tilde{g}_j^{t-1} \big/ \tilde{g}_j^t$ and $\sigma := h_j^{t-1} \big/ h_j^t$. These formulae give discrete analogues of (3). In fact, if we introduce a proper continuous limit [19], then these superposition formulae reduce to (3). We note that (8) and (9) are in the same form except for the parameters (wave number in (8) and frequency in (9)).

Let us give an example of the relation among solutions in the superposition formula (8). If we take the vacuum solution $\rho_0 = 1$ and the one-soliton solutions,

$$\rho_{\mathrm{I}} = \frac{\frac{d_1 X_1}{k_1} + \frac{k_1}{d_1 X_1}}{d_1 X_1 + \frac{1}{d_1 X_1}} \tag{10}$$

$$\rho_{\mathrm{II}} = \frac{\frac{d_2 X_2}{k_2} - \frac{k_2}{d_2 X_2}}{d_2 X_2 - \frac{1}{d_2 X_2}} \tag{11}$$

in (8), then ρ gives the regular two-soliton solution,

$$\rho = \frac{\frac{d_1 d_2 X_1 X_2}{k_1 k_2} + A_{12}\frac{k_2 d_1 X_1}{k_1 d_2 X_2} + A_{12}\frac{k_1 d_2 X_2}{k_2 d_1 X_1} + \frac{k_1 k_2}{d_1 d_2 X_1 X_2}}{d_1 d_2 X_1 X_2 + A_{12}\frac{d_1 X_1}{d_2 X_2} + A_{12}\frac{d_2 X_2}{d_1 X_1} + \frac{1}{d_1 d_2 X_1 X_2}}, \tag{12}$$

where $A_{12} := \left(k_1{}^2 k_2{}^2 - 1\right)/\left(k_2{}^2 - k_1{}^2\right)$.

3. Ultradiscrete system

Let us consider an ultradiscrete analogue of the Bäcklund transformation. We rewrite the Bäcklund transformation into the form convenient to ultradiscretize. We first define

$$\phi_j^t := \frac{f_{j-1}^t}{f_j^t}, \qquad \psi_j^t := \frac{g_{j-1}^t}{g_j^t}. \tag{13}$$

In terms of ϕ_j^t and ψ_j^t, the Bäcklund transformations (7) are rewritten as

$$\begin{aligned}(k_N + 1/k_N)^2(1+\gamma)^2\phi_j^{t-1}\psi_j^{t-1} + \left\{(1+2\gamma)^2 - \frac{(k_N + 1/k_N)^2}{(\omega_N + 1/\omega_N)^2}\right\}\phi_j^t\psi_j^t \\ = (k_N + 1/k_N)(1+\gamma)(1+2\gamma)\left(\psi_j^{t-1}\psi_j^t + \phi_j^{t-1}\phi_j^t\right)\end{aligned} \tag{14a}$$

$$\frac{\phi_j^t}{\psi_j^t} = \frac{(\omega_N + 1/\omega_N)(1+2\gamma)\psi_j^{t-1} + (k_N + 1/k_N)\phi_j^{t-1}\prod_{l=-\infty}^{j-1}\left(\phi_l^{t-1}\psi_l^t\right)/\left(\phi_l^t\psi_l^{t-1}\right)}{(\omega_N + 1/\omega_N)(1+2\gamma)\phi_j^{t-1} + (k_N + 1/k_N)\psi_j^{t-1}\prod_{l=-\infty}^{j-1}\left(\phi_l^t\psi_l^{t-1}\right)/\left(\phi_l^{t-1}\psi_l^t\right)}. \tag{14b}$$

Note that (14*a*) is an ordinary difference equation. If we introduce a proper continuous limit, these formulae reduce to (2). It can be easily shown that the $(N-1)$- and N-soliton solutions constructed from determinant (5) actually satisfy (14).

In order to ultradiscretize (14), we set

$$k_i = \mathrm{e}^{K_i/\varepsilon}, \qquad \omega_i = \mathrm{e}^{\Omega_i/\varepsilon}, \qquad \gamma = \mathrm{e}^{\Gamma/\varepsilon}. \tag{15}$$

If we impose $k_i > 1$, then from the dispersion relation (6) we should have $\omega_i < 1/k_i$ (< 1), which gives $K_i > 0$ and $\Omega_i < 0$. Substituting (15) into (6), applying ε log to the both sides of (6) and taking the limit $\varepsilon \to +0$, we find that (6) is reduced to $\Gamma = -2K_i$. On the other hand, if we impose $0 < k_i < 1$, then we have $K_i < 0$, $\Omega_i > 0$ and $\Gamma = 2K_i$. Thus, the dispersion relation in the ultradiscrete system may be written as

$$K_i\Omega_i < 0, \qquad \Gamma = -2|K_i| < 0. \tag{16}$$

This relation means that any K_i only depends on the system parameter Γ. Instead, Ω_i's are arbitrary constants, which play the roles of soliton parameters. If we put $\phi_j^t = \mathrm{e}^{\Phi_j^t/\varepsilon}$ and

$\psi_j^t = \mathrm{e}^{\Psi_j^t/\varepsilon}$ and take a limit similar to the above, (14) is reduced to the Bäcklund transformation between Φ_j^t and Ψ_j^t,

$$\max\left[\Phi_j^{t-1} + \Psi_j^{t-1} + |K_N|, \Phi_j^t + \Psi_j^t - |K_N|\right] = \max\left[\Phi_j^{t-1} + \Phi_j^t, \Psi_j^{t-1} + \Psi_j^t\right] \tag{17a}$$

$$\Phi_j^t - \Psi_j^t = \max\left[\Psi_j^{t-1} + |\Omega_N|, |K_N| + \Phi_j^{t-1} + \sum_{l=-\infty}^{j-1}\left(\Phi_l^{t-1} - \Psi_l^{t-1}\right) - \sum_{l=-\infty}^{j-1}\left(\Phi_l^t - \Psi_l^t\right)\right]$$
$$- \max\left[\Phi_j^{t-1} + |\Omega_N|, |K_N| + \Psi_j^{t-1} + \sum_{l=-\infty}^{j-1}\left(\Phi_l^t - \Psi_l^t\right) - \sum_{l=-\infty}^{j-1}\left(\Phi_l^{t-1} - \Psi_l^{t-1}\right)\right]. \tag{17b}$$

Since (14) are discrete analogues of (2), these formulae are the ultradiscrete analogues of the Bäcklund transformation. It is to be noted that if we put $f_j^t = \mathrm{e}^{F_j^t/\varepsilon}$ and $g_j^t = \mathrm{e}^{G_j^t/\varepsilon}$, the ultradiscrete analogues of (7) are written as

$$F_j^t + G_{j-1}^{t-1} = \max\left[F_{j-1}^t + G_j^{t-1} - |K_N|, F_j^{t-1} + G_{j-1}^t - |\Omega_N|\right] \tag{18a}$$

$$F_{j-1}^{t-1} + G_j^t = \max\left[F_{j-1}^t + G_j^{t-1} - |\Omega_N|, F_j^{t-1} + G_{j-1}^t - |K_N|\right]. \tag{18b}$$

It is possible to give the ultradiscrete soliton solutions by ultradiscretizing determinant (5). We set $c_i = (-1)^{i+1}/d_i$ and replace k_i, ω_i by (15) and d_i by $e^{\delta_i/\varepsilon}$ in (5), respectively. Then at the limit $\varepsilon \to +0$ the vacuum solution $f_j^t = 1$ is reduced to $F_j^t = 0$ and the one-soliton solution $f_j^t = d_1 X_1 + \frac{1}{d_1 X_1}$ is reduced to

$$F_j^t = \max[-K_1 j - \Omega_1 t - \delta_1, K_1 j + \Omega_1 t + \delta_1]. \tag{19}$$

Similarly, we have the ultradiscrete two-soliton solution,

$$\begin{aligned} F_j^t = \max\big[&-(K_1 + K_2)j - (\Omega_1 + \Omega_2)t - \delta_1 - \delta_2,\\ &(K_1 - K_2)j + (\Omega_1 - \Omega_2)t + \delta_1 - \delta_2 + 2K_1,\\ &(K_2 - K_1)j + (\Omega_2 - \Omega_1)t - \delta_1 + \delta_2 + 2K_1,\\ &(K_1 + K_2)j + (\Omega_1 + \Omega_2)t + \delta_1 + \delta_2\big], \end{aligned} \tag{20}$$

and so on. Note that (13) give the relations

$$\Phi_j^t = F_{j-1}^t - F_j^t, \qquad \Psi_j^t = G_{j-1}^t - G_j^t. \tag{21}$$

By employing these relations, the soliton solutions Φ_j^t and Ψ_j^t satisfying (17) are constructed from a given F_j^t and G_j^t, respectively.

Finally, we study the Bäcklund transformation for the BBS. Let us introduce dependent variables,

$$b_j^t := \frac{\phi_j^t}{\phi_j^{t-1}}, \qquad b_j'^t := \frac{\psi_j^t}{\psi_j^{t-1}}. \tag{22}$$

Both of them obey a discrete analogue of the KdV equation [15],

$$b_j^{t+1} = \frac{1+\gamma}{\gamma b_j^t + \prod_{l=-\infty}^{j-1}\left(b_l^{t+1}/b_l^t\right)}. \tag{23}$$

Rewriting (14), we find that the Bäcklund transformation between b_j^t and $b_j'^t$ is given by

$$(k_N + 1/k_N)^2(1+\gamma)^2 + \left\{(1+2\gamma)^2 - \frac{(k_N + 1/k_N)^2}{(\omega_N + 1/\omega_N)^2}\right\} b_j^t b_j'^t$$
$$= (k_N + 1/k_N)(1+\gamma)(1+2\gamma)\left(b_j'^t \prod_{m=-\infty}^{t-1} \frac{b_j'^m}{b_j^m} + b_j^t \prod_{m=-\infty}^{t-1} \frac{b_j^m}{b_j'^m}\right) \tag{24a}$$

$$\frac{b_j^t}{b_j'^t} = \frac{(\omega_N + 1/\omega_N)(1+2\gamma)\prod_{m=-\infty}^{t-1} b_j'^m/b_j^m + (k_N + 1/k_N)\prod_{l=-\infty}^{j-1} b_l'^t/b_l^t}{(\omega_N + 1/\omega_N)(1+2\gamma)\prod_{m=-\infty}^{t-1} b_j^m/b_j'^m + (k_N + 1/k_N)\prod_{l=-\infty}^{j-1} b_l^t/b_l'^t}. \tag{24b}$$

If we put $b_j^t = \mathrm{e}^{B_j^t/\varepsilon}, b_j'^t = \mathrm{e}^{B_j'^t/\varepsilon}$ and take the limit $\varepsilon \to +0$, we have

$$B_j^t = \Phi_j^t - \Phi_j^{t-1}, \qquad B_j'^t = \Psi_j^t - \Psi_j^{t-1} \tag{25}$$

from (22) and

$$B_j^{t+1} = \min\left[-\Gamma - B_j^t, \sum_{l=-\infty}^{j-1}\left(B_l^t - B_l^{t+1}\right)\right] \tag{26}$$

from (23). Furthermore, we obtain the ultradiscrete analogues of (24),

$$\max\left[|K_N|, B_j^t + B_j'^t - |K_N|\right]$$
$$= \max\left[B_j'^t + \sum_{m=-\infty}^{t-1}\left(B_j'^m - B_j^m\right), B_j^t + \sum_{m=-\infty}^{t-1}\left(B_j^m - B_j'^m\right)\right] \tag{27a}$$

$$B_j^t - B_j'^t = \max\left[|\Omega_N| + \sum_{m=-\infty}^{t-1}\left(B_j'^m - B_j^m\right), |K_N| + \sum_{l=-\infty}^{j-1}\left(B_l'^t - B_l^t\right)\right]$$
$$- \max\left[|\Omega_N| + \sum_{m=-\infty}^{t-1}\left(B_j^m - B_j'^m\right), |K_N| + \sum_{l=-\infty}^{j-1}\left(B_l^t - B_l'^t\right)\right]. \tag{27b}$$

By employing (25) we may construct soliton solutions of the BBS from F_j^t and G_j^t through (21).

Let us give examples of relating the soliton solutions through the Bäcklund transformation. We consider the case of $\Gamma = -1$. Then we have the BBS with the capacity of box 1 and the parameters $K_i = \pm 1/2$. Hereafter, we only take $K_i = 1/2$ for simplicity. If we give the vacuum solution $G_j^t = 0$ and a soliton parameter $\Omega_1 = -1$ in (18), then we have

$$F_j^t = \max\left[-\tfrac{1}{2}j + t - \delta_1, \tfrac{1}{2}j - t + \delta_1\right]. \tag{28}$$

The one-soliton solution B_j^t constructed from (28) describes a soliton with amplitude 2 (see figure 2). Next, giving (28) as G_j^t and a new soliton parameter $\Omega_2 = -3/2$ in (18), we have

$$F_j^t = \max\left[-j + \tfrac{5}{2}t - \delta_1 - \delta_2, \tfrac{1}{2}t + \delta_1 - \delta_2 + 1, -\tfrac{1}{2}t - \delta_1 + \delta_2 + 1, j - \tfrac{5}{2}t + \delta_1 + \delta_2\right]. \tag{29}$$

The two-soliton solution B_j^t constructed from (29) describes the soliton interaction between the soliton with amplitude 2 and a new soliton with amplitude 3 (see figure 3).

$$\rightarrow j$$

```
0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0
```

$$\downarrow t$$

Figure 2. One-soliton solution B_j^t constructed from (28).

$$\rightarrow j$$

```
1 1 1 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 1 1 1 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 1 1 0 0 1 1 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 1 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 1 1 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 1 1 1
```

$$\downarrow t$$

Figure 3. Two-soliton solution B_j^t constructed from (29).

4. Concluding remarks

We have introduced the Bäcklund transformation for the dKdV equation in the bilinear form and shown the way of constructing soliton solutions by the transformation. Moreover, we have presented two types of superposition formulae for the dKdV equation, which are in rather symmetric form.

We have presented the ultradiscrete analogue of the Bäcklund transformation and discussed the construction of ultradiscrete soliton solutions through the transformation. Furthermore, we have given the Bäcklund transformation for the BBS and shown examples of its soliton solutions related through the transformation. However, a difficulty arises in ultradiscretizing the superposition formula, since the singular (nonpositive definite) solution (11) appears in the discrete superposition formula (8) as we have shown in section 2.

It is well known that in the BBS any state consists only of solitons. The close connection between the Bäcklund transformation and the inverse scattering method is also well known for the KdV equation. It is an interesting problem to investigate initial value problems of the ultradiscrete KdV equation by applying our results.

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